A Stable Algorithm for Modeling Lumped Circuit Source Across Multiple FDTD Cells

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***Abstract*— Incorporating lumped-circuit elements into finite-difference time-domain (FDTD) simulation greatly enhances the FDTD method's capability to model both distributed structures and lumped circuitries. The multiple-cell formulation proposed in [1] is only numerically stable for modeling structures with large permittivity. A stable algorithm for modeling material with both large and small permittivity is presented in this letter. Verification data is given together with detailed derivations.**

***Index Terms*—FDTD, multiple-cell FDTD source, stability.**

I. Introduction

THE importance of incorporating lumped-circuit elements into finite-difference time-domain simulation has long been recognized. Sui *et al.* [2] reported a lumped-element formulation for including lumped devices in two-dimensional simulation. The one-cell formulation has been used for modeling digital circuits, active antenna, and some other structures [3].

Recently Durney *et al.* [1] extended the one-cell lumped-circuit formulation into a multiple-cell formulation. The formulation works well for modeling structures with large permittivity but shows unstable operations for structures with small permittivity. The instability problem becomes more severe when spanning range of the multiple cell source is increasing. A stable algorithm based on [1] is presented in this letter. Verification data is given together with detailed derivations.

II. Algorithm

The formulation for incorporating lumped-circuit elements into finite-difference time-domain simulation is based on the integral form of the Ampere's Law [2]

$$\oint_C \vec{H}\cdot d\vec{l} = \iint_S \sigma\vec{E}\cdot d\vec{s} + \frac{\partial}{\partial t}\iint_S \epsilon\vec{E}\cdot d\vec{s} + I_c \qquad (1)$$

where I_c is the current flowing through the lumped-circuit elements.

For the convenience of comparison, all terminologies used by Durney and colleagues [1] are retained here. Formulation for lumped element source in y-direction in a Cartesian coordinate system is presented here. The algorithm for x and z directions can, by analogy, be easily derived. Our algorithm follows [1] until the step for treatment of the voltage across the multiple FDTD cell, i.e., in [1, eq. (4)]:

$$V_c = -(E_{y,i+1,j+1,k} + E_{y,i+1,j,k} + E_{y,i+1,j-1,k})\Delta y. \qquad (2)$$

In deriving updating equation for $E^{n+1}_{y,i+1,j,k}$, only one term, $E_{y,i+1,j,k}$, is time averaged in [1] in order to obtain an explicit updating equation. Unfortunately this is proven to be not sufficient for stable operations, as shown in next section, of multiple-cell lumped-circuit elements.

To avoid instability problem, *all E_y terms on the right-hand side of (2) have to be time averaged*. Written in finite-difference form, (1) becomes

$$\begin{aligned}&(H^n_{z,i,j,k} - H^n_{z,i+1,j,k})\Delta z + (H^n_{x,i+1,j,k+1} - H^n_{x,i+1,j,k})\Delta x \\ &\quad - \frac{V^n_s}{R_s} - \frac{\sum_{m=n_1}^{n_2}(E^{n+1}_{y,i+1,m,k} + E^n_{y,i+1,m,k})\Delta y}{2R_s} \\ &\quad = \left(\frac{\sigma}{2} + \frac{\epsilon}{\Delta t}\right)E^{n+1}_{y,i+1,j,k}\Delta x\Delta z \\ &\quad + \left(\frac{\sigma}{2} - \frac{\epsilon}{\Delta t}\right)E^n_{y,i+1,j,k}\Delta x\Delta z \end{aligned} \qquad (3)$$

where $j = n_1 \sim n_2$ is the index range which the multiple-cell lumped source occupies. There are totally $n_2 - n_1 + 1$ equations involved. It can be seen that the E^{n+1}_y terms in the source region are inter-related and it would be not easy to get an explicit iteration equation for those terms. Reformulating (3) results in following linear equations for E^{n+1}_y:

$$[C_{pq}][E^{n+1}_{y,i+1,p,k}] = [B_p] \qquad (4)$$

where $p, q = n_1 \sim n_2$,

$$\begin{aligned}B_p = &\left(\frac{\sigma}{2} - \frac{\epsilon}{\Delta t}\right)E^n_{y,i+1,p,k}\Delta x\Delta z \\ &+ (H^n_{z,i,p,k} - H^n_{z,i+1,p,k})\Delta z \\ &+ (H^n_{x,i+1,p,k+1} - H^n_{x,i+1,p,k})\Delta x\end{aligned}$$

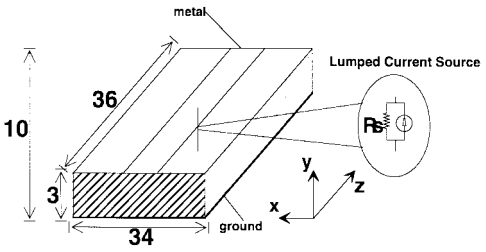

Fig. 1. The structure for stability testing. The whole structure is enclosed by perfect conducting metal. The shaded region is substrate with dielectric constant ϵ_r and conductivity σ. The enlarged drawing on the right-hand side shows detail of the lumped current source across three FDTD cells.

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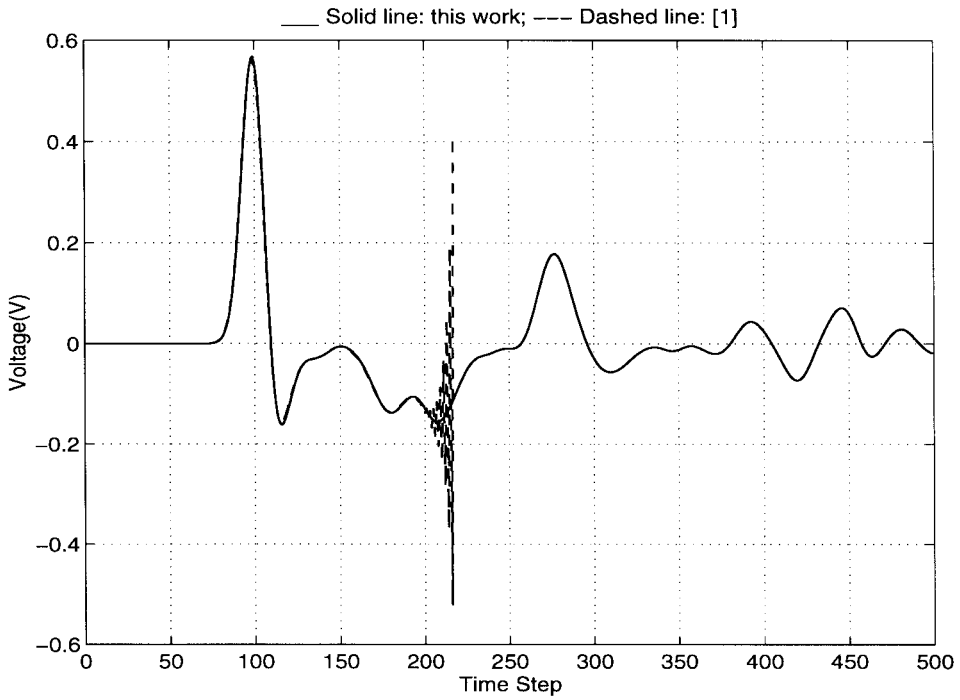

Fig. 2. Stability comparison of the new algorithm in this work with the formulation in [1] for a typical case with $\epsilon = 1.0$, $\sigma = 0.0$ S/m. The voltage compared is the voltage across the three cells lumped current source. Dashed line indicates data based on [1]; solid line indicates data based on this work. Only data from early steps are shown.

$$-\frac{V_s^n}{R_s} - \frac{\sum_{m=n_1}^{n_2} E_{y,i+1,m,k}^n \Delta y}{2R_s} \tag{5}$$

$$C_{pq} = \begin{cases} \frac{\sigma}{2} + \frac{\epsilon}{\Delta t}, & p \neq q \\ \frac{\sigma}{2} + \frac{\epsilon}{\Delta t} + \frac{\Delta y}{2R_s} & p = q. \end{cases} \tag{6}$$

For finite source resistance R_s, determinant of matrix $[C]$ is always positive. It indicates that there is a unique solution to (4). The equation can be solved by usual lower and upper triangular matrix (LU) decomposition and back-substitution method [4]. In the case that size of $[C]$ becomes very large, LU decomposition may become inaccurate and other methods, for example, the singular value decomposition method may have to be used to solve the above equation [4]. For most of the engineering problems, the LU decomposition method should be quite sufficient. The LU decomposition for matrix $[C]$ only needs to be computed once. Therefore overall computational loading is comparable to that in [1].

The next section presents the verification for the new algorithm. Comparison is done with formulations in [1].

III. Verification

General analytical stability criterion is not readily available for checking the new algorithm. Instead, a numerical experiment is done for the algorithm by checking different combinations of material permittivity and conductivity in FDTD simulation domain. To exclude any possible unstable effects arising from absorbing boundary conditions on test simulation and also to check the algorithm in a worst case situation, the test FDTD domain has been enclosed by a perfect conducting (PEC) metal wall. Considering possible roundoff error on the PEC-enclosed structure, a separate test with Mur first-order absorbing boundary condition [3] has also been done for surrounding walls except the ground plane. Same observation has been drawn as from the PEC enclosure case. Only test results with PEC enclosure case are presented below.

The test structure is shown in Fig. 1. FDTD domain size is $34 \times 10 \times 36$ in x, y, and z directions, respectively. Cell dimensions are $\Delta x = 0.6$ mm, $\Delta y = 0.2$ mm, and $\Delta z = 0.4$ mm. Time step is $\Delta t = 0.570\,68$ ps. A y-directed lumped current source across three FDTD cells is put under the metal strip with four-cells width. Substrate under metal strip has ϵ_r as dielectric constant and σ as conductivity. The current source launches a Gaussian pulse with a width of $14\Delta t$ and a peak magnitude of 20 mA. Internal resistance of the source is 50 Ω. Each simulation is running up to 16 000 time steps.

To ensure that both algorithms have been implemented free of any bugs, an accuracy test has been done for both algorithms in a typical case with $\epsilon_r = 2.0$ and $\sigma = 10^{-5}$ S/m for the substrate. In this case the formulation in [1] is stable. Recorded voltages and currents across source region show almost indistinguishable difference between the two algorithms. Therefore implementations for both algorithms are justified.

For stability test, different combinations of ϵ_r and σ have been checked and compared for the two algorithms, with ϵ_r varying from 1.0 through 80.0 and σ from 0.0 through 10^{+7} S/m. First, the formulation proposed in [1] is tested under all the conditions mentioned above. Its stability performance is tabulated in Table I with $\oplus$ indicating stable operation and $\ominus$ unstable behavior. Number behind $\ominus$ shows the time step when data starts to blow up and instability is recorded. It can be seen that the formulation proposed in [1] works well with large material ϵ_r but fails for ϵ_r close to 1.0. As shown in the

TABLE I
STABILITY TABLE FOR THE FORMULATION IN [1] WHEN SOURCE OCCUPYING THREE FDTD CELLS. ⊕ INDICATES STABLE OPERATION AND ⊖ UNSTABLE BEHAVIOR. NUMBERS BEHIND ⊖ INDICATE THE TIME STEPS WHEN DATA STARTS TO BLOW UP

	ϵ_r=1	1.2	2	4	10	20	40	80
σ=0	⊖200	⊖2000	⊕	⊕	⊕	⊕	⊕	⊕
10^{-5}	⊖210	⊖2000	⊖	⊖				
10^{-3}	⊖210	⊖2000	⊕		⊕			
10^{-1}	⊖210	⊖2000	⊕			⊕		
10^{+1}	⊖210	⊖1700	⊕					
10^{+3}	⊖1700	⊕	⊕				⊕	
10^{+5}	⊖	⊕	⊕					
10^{+7}	⊕	⊕	⊖					⊕

TABLE II
STABILITY TABLE FOR THE FORMULATION IN [1] WHEN SOURCE OCCUPYING TEN FDTD CELLS. ⊕ INDICATES STABLE OPERATION AND ⊖ UNSTABLE BEHAVIOR. NUMBERS BEHIND ⊖ INDICATE THE TIME STEPS WHEN DATA STARTS TO BLOW UP

	ϵ_r=1	1.2	2	4	10	20	40	80
σ=0	$\ominus_{21}$	$\ominus_{35}$	$\ominus_{81}$	$\ominus_{165}$	⊕	⊕	⊕	⊕
10^{-5}	$\ominus_{21}$	$\ominus_{35}$	$\ominus_{81}$	$\ominus_{165}$	⊕			
10^{-3}	$\ominus_{21}$	$\ominus_{35}$	$\ominus_{81}$	$\ominus_{165}$	⊕			
10^{-1}	$\ominus_{21}$	$\ominus_{35}$	$\ominus_{81}$	$\ominus_{165}$	⊕	⊕		
10^{+1}	$\ominus_{21}$	$\ominus_{49}$	$\ominus_{89}$	$\ominus_{170}$	⊖			
10^{+3}	$\ominus_{220}$	$\ominus_{220}$	$\ominus_{255}$	$\ominus_{540}$	⊕		⊕	
10^{+5}	$\ominus_{7000}$	$\ominus_{7200}$	$\ominus_{8800}$	⊕	⊕			
10^{+7}	⊕	⊕	⊕	⊕	⊕			⊕

first column in Table I in the case $\epsilon_r = 1.0$, the formulation in [1] is not stabilized until substrate conductivity σ is increased to 10^{+5} S/m! Complete test shows that the unstable range for the formulation in [1] is $\epsilon_r = 1 \sim 1.2$ in the case $\sigma = 0.0$.

The same set of stability tests as those for [1] is carried out for the new algorithm proposed in this work. The new algorithm performs very stable for all testing cases. A comparison of recorded voltages across source regions is given in Fig. 2 for the two algorithms in the case $\epsilon_r = 1.0$ and $\sigma = 0.0$ S/m. It can be seen that the algorithm in [1] becomes unstable soon after the excitation pulse is launched out. The new algorithm proposed in this work functions very stable.

The exact reason why the formulation in [1] fails for low material permittivity is not yet known. It is clear that there exist some ambiguities on determining the exact time moment of V_c when it is calculated from (4) in [1]. The most distinguishable difference between the new algorithm in this work and that in [1] lies in whether all E_y^{n+1} terms are time averaged or not. This definitely has made the difference to ensure the new algorithm stable.

To check effect of spanning range of the excitation source on algorithm stability, another test has been carried out for the two algorithms by increasing spanning range of the excitation from three cells to ten cells. Accordingly, the FDTD domain size is changed from $34 \times 10 \times 36$ to $34 \times 20 \times 36$. The stability performance is listed in Table II with the same notations as in Table I. It was found that the unstable range for the algorithm in [1] has expanded to $\epsilon_r = 1 \sim 4$ in the case $\sigma = 0.0$. This is because more errors are generated in V_c while the number of cells in the source region is increased. The new algorithm functions stable for all simulation tests.

IV. CONCLUSION

A stable algorithm for modeling lumped-circuit source across multiple FDTD cells has been presented and verified in this letter. The algorithm is directly applicable to lumped-circuit resistors and can be readily extended to other lumped-circuit elements. The new algorithm would greatly enhance the FDTD method's capability to model both distributed structures and lumped circuitries. Once again it has been verified that in order to ensure stable FDTD simulations all the electric field terms in lumped-element region have to be time averaged for multiple cell formulations. Future study would include a theoretical verification of the stability performance of the new algorithm.

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